

Gauged/Massive Supergravities in Diverse Dimensions

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Abstract

We show how massive/gauged maximal supergravities in $11 - n$ dimensions with $SO(n - l, l)$ gauge groups (and other non-semisimple subgroups of $Sl(n, \mathbb{R})$) can be systematically obtained by dimensional reduction of “massive 11-dimensional supergravity”. This series of massive/gauged supergravities includes, for instance, Romans’ massive $N = 2A, d = 10$ supergravity for $n = 1$, $N = 2, d = 9$ $SO(2)$ and $SO(1, 1)$ gauged supergravities for $n = 2$, and $N = 8, d = 5$ $SO(6 - l, l)$ gauged supergravity. In all cases, higher p -form fields get masses through the Stückelberg mechanism which is an alternative to self-duality in odd dimensions.

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1 Introduction

Massive/gauged supergravities are very interesting theories which have become fashionable due to the relation between the presence in their Lagrangians of mass/gauge coupling parameters with the existence of domain-wall-type solutions and the holographic relation between (in general non-conformal) the gauge field theories that live on the domain wall and the superstring/supergravity theories that live in the bulk [1, 2] (for a review, see Ref. [3]).

These supergravity theories appear in the literature in essentially 3 different ways in the compactification of ungauged (massless) supergravities:

1. In compactifications in non-trivial internal manifolds (particularly Freund-Rubin-type [4] spontaneous compactifications on spheres). Some notable examples are the S^7 [5] compactification of 11-dimensional supergravity that is supposed to give the $SO(8)$ -gauged $N = 8, d = 4$ supergravity, the S^4 compactification of 11-dimensional supergravity [6] that gives [7] the $SO(5)$ -gauged $N = 4, d = 7$ theory, and the S^5 compactification of $N = 2B, d = 10$ supergravity [8, 9, 10] that gives the $SO(6)$ -gauged $N = 8, d = 5$ supergravity theory.
2. In Scherk-Schwarz generalized dimensional reductions [11], in which the global symmetry is geometrical or non-geometrical.

Examples in which a global geometrical $SU(2)$ symmetry has been used to obtain a gauged/massive supergravity are Salam & Sezgin's compactification of 11-dimensional supergravity to obtain $SU(2)$ -gauged $N = 4, d = 8$ supergravity [12], and Chamseddine and Volkov's obtention of $SU(2) \times SU(2)$ -gauged $N = 4, d = 4$ supergravity [13, 14] and Chamseddine and Sabra's obtention of and $SU(2)$ -gauged $N = 2, d = 7$ supergravity [15] from $N = 1, d = 10$ supergravity in both cases.

An example in which a global symmetry of non-geometrical origin is used to obtain a gauged/massive supergravity by generalized dimensional reduction is the obtention of massive $N = 2, d = 9$ supergravity from $N = 2B, d = 10$ supergravity exploiting the axion's shift symmetry [16]. If one exploits the full global $Sl(2, \mathbb{R})$ symmetry of the $N = 2B, d = 10$ theory one obtains a 3-parameter family of supergravity theories [17] (see also [18]) some of which are gauged supergravities [19]. From the string theory point of view, the three parameters take discrete values which must be considered equivalent when they are related by an $Sl(2, \mathbb{Z})$ duality transformation (i.e. when they belong to the same conjugacy class) and they describe the low-energy limit of the same string theory [20]. There is, actually, an infinite number of $Sl(2, \mathbb{Z})$ conjugacy classes and for each of them one gets a massive/gauged supergravity with either $SO(2)$, $SO(1, 1)$ or no gauge group [21]³. We will take a closer look later to this $Sl(2, \mathbb{Z})$ family of theories.

3. In compactifications with non-trivial p -form fluxes (see e.g. [24]).

These three instances are not totally unrelated. To start with, compactifications with fluxes can be understood as non-geometrical Scherk-Schwarz reductions in which the global

³These gauged supergravity theories have been constructed by direct gauging in Ref. [22]. Further, very recently, new gauged $N = 2, d = 9$ supergravities constructed by Scherck-Schwarz generalized dimensional reduction have been presented in Ref. [23]. One of them has a 2-dimensional non-Abelian gauge group.

symmetry exploited is the one generated by p -form “gauge” transformations with constant parameters. The axion shift symmetry can be understood as the limit case $p = -1$ and can be used in compactifications on circles. Higher p -form fluxes can only be exploited in higher-dimensional internal spaces that can support them. On the other hand, the geometrical Scherk-Schwarz compactifications used by Salam & Sezgin and Chamseddine, Volkov and Sabra could be understood as compactification on the $SU(2)$ group manifold S^3 although one would expect a gauge group $SO(4) \sim SU(2) \times SU(2)$ since this is the isometry group of the S^3 metric used.

Finally, the Freund-Rubin spontaneous sphere compactifications are compactifications on a brane background (more precisely, in a brane’s near-horizon geometry) and there is a net flux of the form associated to the brane, while the non-geometrical Scherk-Schwarz compactifications can also be seen as compactifications on a $(d-3)$ -brane background [17], in which the brane couples to the $(d-2)$ -form potential dual to the scalar.

Historically, almost all the gauged/massive theories we just discussed had been constructed by gauging or mass-deforming known ungauged/massless theories⁴ the only exception being the $N = 2, d = 9$ theories that, in principle, could have been constructed in that way as well.

A crucial ingredient in the gauging of some of the higher-dimensional supergravities with p -form fields transforming under the global symmetry being gauged is that these fields must be given a mass whose value is related by supersymmetry to the gauge coupling parameter: if the p -form fields remained massless, they should transform simultaneously under their own massless p -form gauge transformations (to decouple negative-norm states) and under the new gauge transformations, which is impossible. A mass term eliminates the requirement of massless p -form gauge invariance but introduces another problem, because the number of degrees of freedom of the theory should remain invariant. In the cases of the $SO(5)$ -gauged $N = 4, d = 7$ and the $SO(6)$ -gauged $N = 8, d = 5$ theories this was achieved by using the “self-duality in odd dimensions” mechanism [33, 34] which we will explain later on.

The need to introduce mass parameters together with the gauge coupling constant is one of the reasons why we call these theories gauged/massive supergravities. In some cases no mass parameters will be needed in the gauging and in some others no gauge symmetry will be present when the mass parameters are present but they can nevertheless be seen as members of the same class of theories. Another reason is that in many cases the gauge parameter has simultaneously the interpretation of gauge coupling constant and mass of a domain-wall solution of the theory⁵ that can correspond to the near-horizon limit of some higher-dimensional brane solution apart from that of the mass of a given field in the Lagrangian.

The gauging and mass-deformation procedures are very effective tools to produce gauged theories in a convenient form but hide completely their possible higher-dimensional or string/M-theoretical origin. In fact, there are many gauged/massive supergravity theories whose string- or M-Theoretical origin is still unknown, which, in supergravity language

⁴The $SO(8)$ -gauged $N = 8, d = 4$ supergravity was constructed in Ref. [25, 26], the $SO(5)$ -gauged $N = 4, d = 7$ supergravity in Ref. [27], the $SO(6)$ -gauged $N = 8, d = 5$ supergravity in Refs. [28, 29], the $SU(2) \times SU(2)$ -gauged $N = 4, d = 4$ supergravity in Ref. [30] and the $SU(2)$ -gauged $N = 2, d = 7$ supergravity in Ref. [31]. Romans’ massive $N = 2A, d = 10$ supergravity was obtained by a deformation of the massless theory [32].

⁵Anti-de Sitter spacetime can also be interpreted as a domain-wall solution, its mass being related to the cosmological constant.

means that we do not know how to obtain them by some compactification procedure from some higher-dimensional (ungauged/massless) theory. In some cases, it is known how to obtain it from $N = 2B, d = 10$ supergravity but not from the $N = 2A, d = 10$ or 11-dimensional supergravity. A notorious example is Romans' massive $N = 2A, d = 10$ supergravity [32] that cannot be obtained from standard 11-dimensional supergravity (a theory that cannot be deformed to accommodate a mass parameter preserving 11-dimensional Lorentz invariance [35, 36, 37]) by any sort of generalized dimensional reduction, but there are many more. Let us review some other examples:

1. The $Sl(2, \mathbb{Z})$ family of $N = 2, d = 9$ gauged/massive supergravities are obtained by Scherk-Schwarz reduction of the $N = 2B, d = 10$ theory, but it is not known how to obtain them from standard 11-dimensional or $N = 2A, d = 10$ supergravity.

In these theories, the $Sl(2, \mathbb{R})$ doublet of 2-form potentials gets masses through the Stückelberg mechanism.

2. The massless $N = 4, d = 8$ supergravity contains two $SU(2)$ triplets of vector fields. The two triplets are related by $Sl(2, \mathbb{R})$ S-duality transformations. It should be possible to gauge $SU(2)$ using as $SU(2)$ gauge fields any of the two triplets. If we gauged the triplet of Kaluza-Klein vectors, we would get the theory that Salam & Sezgin obtained by Scherk-Schwarz reduction of 11-dimensional supergravity. It is not known how to derive from standard 11-dimensional supergravity the “S-dual” theory that one would get gauging the other triplet, that comes from the 11-dimensional 3-form.

In these two theories, the $SU(2)$ triplet of 2-form potentials gets masses through the Stückelberg mechanism, eating the 3 vectors that are not $SU(2)$ -gauged.

3. The $SO(5)$ -gauged $N = 4, d = 7$ supergravity theory is also just a particular member of the family of $SO(5 - l, l)$ gauged $N = 4, d = 7$ supergravities constructed in Ref. [38]. The 11-dimensional origin of the $SO(5)$ theory is well understood, but not that of the theories with non-compact gauge group.
4. The $SO(6)$ -gauged $N = 8, d = 5$ supergravity theory is also just a particular member of the family of $SO(6 - l, l)$ gauged $N = 8, d = 5$ supergravities constructed in Refs. [28, 29]. Again, while the $N = 2B, d = 10$ origin of the $SO(6)$ theory is well understood, its 11-dimensional origin and the higher-dimensional origin of the theories with non-compact gauge groups is unknown.
5. Essentially the same can be said about the $SO(8)$ -gauged $N = 8, d = 4$ supergravity theory since it is possible to generate from it by analytical continuation theories with non-compact groups $SO(8 - l, l)$ [39] whose higher-dimensional origin is also unknown.

In the search for an 11-dimensional origin of Romans' massive $N = 2A, d = 10$ theory a “massive 11-dimensional supergravity” was proposed in Ref. [40]. This theory is a deformation of the standard 11-dimensional supergravity that contains a mass parameter and, to evade the no-go theorem of Refs. [35, 36, 37], a Killing vector in the Lagrangian that effectively breaks the 11-dimensional Lorentz symmetry to the 10-dimensional one even if the theory is formally 11-dimensional covariant. Standard dimensional reduction in the direction of the Killing vector gives the Lagrangian of Romans' theory.

This theory was little more than the straightforward uplift of Romans' but it could be generalized to one with n Killing vectors and a symmetric⁶ $n \times n$ mass matrix \mathbf{Q}^{mn} [17]. The reduction of the $n = 2$ theory in the direction of the two Killing vectors turns out to give all the $SO(2-l, l)$ -gauged $N = 2, d = 9$ supergravities obtained by Scherk-Schwarz reduction from $N = 2B, d = 10$ supergravity [17, 18]: each of these theories is determined by a traceless 2×2 matrix m^m_n of the $sl(2, \mathbb{R})$ Lie algebra which is related to the symmetric mass matrix \mathbf{Q}^{mn} by

$$\mathbf{Q}^{mn} = \eta^{mp} \mathbf{m}_p^n, \quad \eta_{mn} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta^{mn} = -\eta_{mn}.$$

The reduction of the $n = 3$ theory gives the “S-dual” $SU(2)$ -gauged $N = 4, d = 8$ theory mentioned above when we make the choice $\mathbf{Q}^{mn} = g\delta^{mn}$ [41] but can also give the theories with non-compact gauge group $SO(2, 1)$ if we choose $\mathbf{Q} = g \text{diag}(+ + -)$. Singular \mathbf{Q} s give rise other 3-dimensional non-semisimple gauge groups and massive/ungauged supergravities, as we are going to show.

In this paper we are going to study these and other gauged/massive theories obtained by dimensional reduction of “massive 11-dimensional supergravity” with n Killing vectors [40, 17]. Generically, the theories obtained in this way are $(11-n)$ -dimensional supergravity theories with 32 supercharges determined by a mass matrix \mathbf{Q}^{mn} . They are *covariant* under global $Sl(n, \mathbb{R})$ duality transformations that in general transform \mathbf{Q}^{mn} into the mass matrix of another theory⁷ of the same family.

The subgroup of $Sl(n, \mathbb{R})$ that preserves the mass matrix is a symmetry of the theory and at the end it will be the gauge group. If we use $Sl(n, \mathbb{R})$ transformations and rescalings to diagonalize the mass matrix so it has only $+1, -1, 0$ in the diagonal, it is clear that $SO(n, n-l)$ will be amongst the possible gauge groups and corresponds to a non-singular mass matrix. These theories with non-singular mass matrices have $n(n-1)/2$ vector fields coming from the $\hat{C}_{\mu mn}$ components of the 11-dimensional 3-form and transforming as $SO(n-l, l)$ $l = 0, \dots, n$ gauge vector fields plus n 2-forms with the same origin and n Kaluza-Klein vectors coming from the 11-dimensional metric that transform as $SO(n-l, l)$ n -plets. The n vectors act as Stückelberg fields for the 2-forms which become massive. In this way the theory is consistent with the $SO(n-l, l)$ gauge symmetry.

Finally, all these theories have a scalar potential that contains a universal term of the form

$$\mathcal{V} = -\frac{1}{2}e^{\alpha\varphi} \{[\text{Tr}(\mathbf{Q}\mathcal{M})]^2 - 2\text{Tr}(\mathbf{Q}\mathcal{M})^2\}, \quad (1.1)$$

where \mathcal{M} is a (symmetric) $Sl(n, \mathbb{R})/SO(n)$ scalar matrix, plus, possibly, other terms from the scalars that come from the 3-form. That scalar potentials of this form appears in several gauged supergravities was already noticed in Refs. [12, 43]. The $d = 5$ case is special because $\alpha = 0$. This is related to the invariance of the Lagrangian under the $N = 2B, d = 10$ $Sl(2, \mathbb{R})$ symmetry.

⁶We will discuss later on the possibility of generalizing further the theory by admitting non-symmetric matrices \mathbf{Q}^{mn} .

⁷In all cases we expect the entries of the mass matrix \mathbf{Q}^{mn} to be quantized and take integer values in appropriate units, since they are related to tensions and charges of branes which are quantized in string theory. The duality group is then broken to $Sl(n, \mathbb{Z})$ [42]. Theories related by $Sl(n, \mathbb{Z})$ transformations should be considered equivalent from the string theory point of view. We will take into account these subtleties later.

Some of these theories are known, albeit in a very different form. The case $n = 6$ is particularly interesting: we get $SO(6-l, l)$ -gauged $N = 8, d = 5$ supergravities which were constructed by explicit gauging in Refs. [28, 29], with 15 gauge vectors that originate in the 3-form, 6 Kaluza-Klein vector fields that originate in the metric and give mass by the Stückelberg mechanism to 6 2-forms that come from the 3-form. That is: the field content (but not the couplings nor the spectrum) is the same as that of the ungauged theory that one would obtain by straightforward toroidal dimensional reduction. In fact, the ungauged theory can be recovered by taking the limit $Q \rightarrow 0$ which is non-singular. In Refs. [28, 29] the gauged theories were constructed by dualizing first the 6 vectors into 2-forms that, together with the other 6 2-forms, satisfy self-duality equations [33] and describe also the degrees of freedom of 6 massive 2-forms. In this theory the massless limit is singular and can only be taken after the elimination of the 6 unphysical 2-forms [34].

Thus, we have, presumably, two different versions of the same theory in which the 6 massive 2-forms are described using the Stückelberg formalism or the self-duality formalism. We will try to show the full equivalence between both formulations at the classical level.

Something similar happens in $d = 7$, although we get $SO(4-l, l)$ -gauged theories and in the literature only $SO(5-l, l)$ -gauged theories have been constructed [27, 38].

This paper is organized as follows: in Section 2 we describe the “massive 11-dimensional supergravity”, its Lagrangian and symmetries. In Section 3 we briefly review how for $n = 1$ we recover Romans’ massive $N = 2A, d = 10$ supergravity. In Section 4 we revise how for $n = 2$ we get the $Sl(2, \mathbb{Z})$ family of gauged/massive $N = 2, d = 9$ supergravities and how they are classified by their mass matrix. In Section 5 we study the case $n = 3$ and the gauged/massive $N = 2, d = 8$ supergravities that arise which allows us to describe the general situation for arbitrary n . In Section 6 we study the $n = 6$ case and try to argue that we have obtained an alternative but fully equivalent form of the $SO(6-l, l)$ -gauged $N = 8, d = 5$.

2 Massive 11-Dimensional Supergravity

Massive 11-dimensional supergravity can be understood as a deformation of standard 11-dimensional supergravity [45] that breaks 11-dimensional Lorentz invariance. The bosonic fields of the standard $N = 1, d = 11$ supergravity are the Elfbein and a 3-form potential⁸

$$\left\{ \hat{e}_{\hat{\mu}}^{\hat{a}}, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} \right\}. \quad (2.1)$$

The field strength of the 3-form is

$$\hat{G} = 4 \partial \hat{C}, \quad (2.2)$$

and is obviously invariant under the massless 3-form gauge transformations

$$\delta \hat{C} = 3 \partial \hat{\chi}, \quad (2.3)$$

where $\hat{\chi}$ is a 2-form. The action for these bosonic fields is

$$\hat{S} = \int d^{11} \hat{x} \sqrt{|\hat{g}|} \left[\hat{R} - \frac{1}{2 \cdot 4!} \hat{G}^2 - \frac{1}{(144)^2} \frac{1}{\sqrt{|\hat{g}|}} \hat{e} \hat{G} \hat{G} \hat{C} \right]. \quad (2.4)$$

⁸Our conventions are those of Refs. [17, 41]. In particular, $\hat{\mu}$ (\hat{a}) are curved (flat) 11-dimensional indices and our signature is $(+ - \dots -)$.

Now, we are going to assume that all the fields of this theory are independent of n internal coordinates z^m . This is the standard assumption in toroidal dimensional reductions and, in particular, it means that the metric admits n mutually commuting Killing vectors $\hat{k}_{(n)}$ associated to the internal coordinates by

$$\hat{k}_{(m)}^{\hat{\mu}} \partial_{\hat{\mu}} = \frac{\partial}{\partial z^m} \equiv \partial_m. \quad (2.5)$$

We are also going to introduce an arbitrary symmetric *mass* matrix Q^{mn} . The possibility or need to introduce a more general mass matrix will be discussed later on. With these elements (the Killing vectors and the mass matrix) we are going to deform the massless theory.

First, we construct the *massive gauge parameter* 1-form⁹ $\hat{\lambda}^{(n)}$

$$\hat{\lambda}^{(m)} \equiv -Q^{mn} i_{\hat{k}_{(n)}} \hat{\chi}, \quad (2.6)$$

and, for any 11-dimensional tensor \hat{T} , we define the *massive gauge transformations*

$$\delta_{\hat{\chi}} \hat{T}_{\hat{\mu}_1 \dots \hat{\mu}_r} = -\hat{\lambda}^{(n)}_{\hat{\mu}_1} \hat{k}_{(n)}^{\hat{\nu}} \hat{T}_{\hat{\nu} \hat{\mu}_2 \dots \hat{\mu}_r} - \dots - \hat{\lambda}^{(n)}_{\hat{\mu}_r} \hat{k}_{(n)}^{\hat{\nu}} \hat{T}_{\hat{\mu}_1 \dots \hat{\mu}_{r-1} \hat{\nu}}. \quad (2.7)$$

According to this general rule, the massive gauge transformation of the 11-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ and of any 11-dimensional form of rank r $\hat{A}_{\hat{\mu}_1 \dots \hat{\mu}_r}$ are given by

$$\left\{ \begin{array}{l} \delta_{\hat{\chi}} \hat{g}_{\hat{\mu}\hat{\nu}} = -2\hat{k}_{(m)}_{(\hat{\mu}} \hat{\lambda}^{(m)}_{\hat{\nu})}, \\ \delta_{\hat{\chi}} \hat{A}_{\hat{\mu}_1 \dots \hat{\mu}_r} = (-)^r r \hat{\lambda}^{(n)}_{[\hat{\mu}_1} \left(i_{\hat{k}_{(n)}} \hat{A} \right)_{\hat{\mu}_2 \dots \hat{\mu}_r]}, \end{array} \right. \quad (2.8)$$

which, together, imply

$$\left\{ \begin{array}{l} \delta_{\hat{\chi}} \sqrt{|\hat{g}|} = 0, \\ \delta_{\hat{\chi}} \hat{A}^2 = 0. \end{array} \right. \quad (2.9)$$

However, the 3-form of massive 11-dimensional supergravity does not transform homogeneously under massive gauge transformations, but

$$\delta_{\hat{\chi}} \hat{C} = 3\partial \hat{\chi} - 3\hat{\lambda}^{(n)} i_{\hat{k}_{(n)}} \hat{C}, \quad (2.10)$$

which allows us to see it as a sort of *connection*. It is not surprising that the gauge vectors of the dimensionally reduced gauged/massive theories come from the 11-dimensional 3-form. The massive 4-form field strength is given by

$$\hat{G} = 4\partial \hat{C} - 3Q^{mn} i_{\hat{k}_{(m)}} \hat{C} i_{\hat{k}_{(n)}} \hat{C}, \quad (2.11)$$

and transforms covariantly, according to the above general rule, so

$$\delta_{\hat{\chi}} \hat{G}^2 = 0. \quad (2.12)$$

The action of the proposed *massive 11-dimensional supergravity* then reads

⁹ $i_{\hat{v}} \hat{T}$ denotes the contraction of the last index of the covariant tensor \hat{T} with the vector \hat{v} .

$$\begin{aligned}
\hat{S} = & \int d^{11}\hat{x}\sqrt{|\hat{g}|} \left\{ \hat{R}(\hat{g}) - \frac{1}{2 \cdot 4!} \hat{G}^2 - \hat{K}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{K}^{\hat{\nu}\hat{\rho}\hat{\mu}} + \frac{1}{2} \mathbf{Q}^{mn} d\hat{k}_{(m)} i_{\hat{k}_{(n)}} \hat{C} \right. \\
& + \frac{1}{2} (\mathbf{Q}^{mn} \hat{k}_{(m)}^{\hat{\mu}} \hat{k}_{(n)}^{\hat{\nu}})^2 - (\mathbf{Q}^{mn} \hat{k}_{(m)}^{\hat{\mu}} \hat{k}_{(m)}^{\hat{\nu}})^2 \\
& - \frac{1}{6^4} \frac{\hat{\epsilon}}{\sqrt{|\hat{g}|}} \left[\partial \hat{C} \partial \hat{C} \hat{C} + \frac{9}{8} \mathbf{Q}^{mn} \partial \hat{C} \hat{C} i_{\hat{k}_{(m)}} \hat{C} i_{\hat{k}_{(n)}} \hat{C} \right. \\
& \left. \left. + \frac{27}{80} \mathbf{Q}^{mn} \mathbf{Q}^{pq} \hat{C} i_{\hat{k}_{(m)}} \hat{C} i_{\hat{k}_{(n)}} \hat{C} i_{\hat{k}_{(p)}} \hat{C} i_{\hat{k}_{(q)}} \hat{C} \right] \right\} ,
\end{aligned} \tag{2.13}$$

where we have defined a *contorsion* tensor

$$\hat{K}_{\hat{a}\hat{b}\hat{c}} = \frac{1}{2} \left(\hat{T}_{\hat{a}\hat{c}\hat{b}} + \hat{T}_{\hat{b}\hat{c}\hat{a}} - \hat{T}_{\hat{a}\hat{b}\hat{c}} \right) , \tag{2.14}$$

where the *torsion* tensor is defined by

$$\hat{T}_{\hat{\mu}\hat{\nu}}^{\hat{\rho}} = -\mathbf{Q}^{mn} (i_{\hat{k}_{(m)}} \hat{C})_{\hat{\mu}\hat{\nu}} \hat{k}_{(n)}^{\hat{\rho}} . \tag{2.15}$$

The action is also invariant under massive gauge transformations up to total derivatives.

By construction, this theory is meant to be compactified in the n -dimensional torus parametrized by the coordinates z^m . After that dimensional reduction, the explicit Killing vectors in the Lagrangian disappear and one gets a genuine $(11 - n)$ -dimensional field theory. We will postpone to the last section the discussion of the “true” dimensional nature of the above theory and in the next few sections we will consider it as a systematic prescription to get massive/gauged supergravity theories in $11 - n$ dimensions and we will study these theories for several values of n .

Also by construction, there is a natural action of the group $Gl(n, \mathbb{R})$ in these theories, all the objects carrying m, n indices (including the mass matrix) transforming in the vector representation. The subgroup of $Gl(n, \mathbb{R})$ that preserves the mass matrix will be a symmetry group of the theory.

The gauge invariances of the gauged supergravities that we will obtain are encoded in the 11-dimensional *massive gauge transformations* parametrized by the 1-forms $\hat{\lambda}^{(m)}$. Their dimensional reduction will give rise to further massive gauge transformations parametrized by 1-forms and associated to massive 2-forms $\lambda_\mu^{(m)}$ and will also give rise to (Yang-Mills) gauge transformations parametrized by the scalars $\lambda_n^{(m)}$ where the subindex n corresponds to an internal direction. These scalars exist when there is more than one Killing vector and are antisymmetric in the indices m, n and correspond to orthogonal gauge groups. This is consistent with the fact that the gauge vectors come from the components $\hat{C}_{\mu mn}$ and naturally carry a pair of antisymmetric indices corresponding to the adjoint representation of an orthogonal group.

A few words about the fermions of the theory, which we have so far ignored, are in order. The above theory is a straightforward generalization to arbitrary n of the $n = 2$ case obtained by uplifting of the gauged/massive $N = 2, d = 9$ supergravities constructed in Ref. [17] by non-geometrical Scherk-Schwarz reduction of the $N = 2B, d = 10$ theory. This construction was made for the bosonic sector only, but can be made for the full supergravity Lagrangian, as shown in Ref. [18]. Once the full gauged/massive $N = 2, d = 9$ supergravity is constructed it can be uplifted to $d = 11$ and then generalized to arbitrary n . This was

done for the fermionic supersymmetry transformation rules in [18] and they were found to have the form

$$\begin{aligned} \frac{1}{2}\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} &= \left\{ \hat{\nabla}_{\hat{\mu}}(\hat{\omega} + \hat{K}) + \frac{i}{288} \left[\hat{\Gamma}^{\hat{a}\hat{b}\hat{c}\hat{d}}_{\hat{\mu}} - 8\hat{\Gamma}^{\hat{b}\hat{c}\hat{d}}_{\hat{\mu}}\hat{\hat{e}}^{\hat{a}} \right] \hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}} \right. \\ &\quad \left. - \frac{i}{12}\hat{k}_{(n)\hat{\nu}}\mathbf{Q}^{nm}\hat{k}_{(m)}^{\hat{\nu}}\hat{\Gamma}_{\hat{\mu}} + \frac{i}{2}\hat{k}_{(n)\hat{\mu}}\mathbf{Q}^{nm}\hat{k}_{(m)\hat{\nu}}\hat{\Gamma}^{\hat{\nu}} \right\} \hat{\epsilon}. \end{aligned} \quad (2.16)$$

It should be clear from this discussion that a fully supersymmetric theory is obtained for each value of n . In the next sections we are going to see how known gauged/massive supergravities arise in the dimensional reduction of the above Lagrangian and supersymmetry transformation rules in the direction of the Killing vectors $\hat{k}_{(m)}$.

3 Romans' Massive $N = 2A, d = 10$ Supergravity from $d = 11$

The reduction of the $n = 1$ case in the direction of the unique Killing vector present with the same Kaluza-Klein Ansatz as in the massless case gives Romans' massive $N = 2A, d = 10$ supergravity [40] with the field content (in stringy notation)

$$\{g_{\mu\nu}, \phi, B_{\mu\nu}, C^{(3)}_{\mu\nu\rho}, C^{(1)}_{\mu}, \psi_{\mu}, \lambda\} \quad (3.1)$$

and with a mass parameter¹⁰ m equal to minus the mass matrix $m = -\mathbf{Q}$. Thus, setting for the bosonic fields

$$(\hat{e}_{\hat{\mu}}^{\hat{a}}) = \begin{pmatrix} e^{-\frac{1}{3}\phi}e_{\mu}^a & e^{\frac{2}{3}\phi}C^{(1)}_{\mu} \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix}, \quad (\hat{e}_{\hat{a}}^{\hat{\mu}}) = \begin{pmatrix} e^{\frac{1}{3}\phi}e_a^{\mu} & -e^{\frac{1}{3}\phi}C^{(1)}_a \\ 0 & e^{-\frac{2}{3}\phi} \end{pmatrix}, \quad (3.2)$$

$$\hat{C}_{\mu\nu\rho} = C^{(3)}_{\mu\nu\rho}, \quad \hat{C}_{\mu\nu\bar{z}} = B_{\mu\nu}, \quad (3.3)$$

we get the string-frame bosonic action

$$\begin{aligned} S &= \int d^{10}x \sqrt{|g|} \left\{ e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{2\cdot 3!}H^2 \right] - \left[\frac{1}{2}m^2 + \frac{1}{2\cdot 2!}m^2 (G^{(2)})^2 + \frac{1}{2\cdot 4!} (G^{(4)})^2 \right] \right. \\ &\quad \left. - \frac{1}{144} \frac{1}{\sqrt{|g|}} \epsilon \left[\partial C^{(3)} \partial C^{(3)} B + \frac{1}{2} m \partial C^{(3)} BBB + \frac{9}{80} m^2 BBBBB \right] \right\}, \end{aligned} \quad (3.4)$$

where $G^{(2)}$ and $G^{(4)}$ are the RR 2- and 4-form field strengths

$$G^{(2)} = 2\partial C^{(1)} + mB, \quad G^{(4)} = 4\partial C^{(3)} - 12\partial BC^{(1)} + 3mBB, \quad (3.5)$$

and H is the NSNS 3-form field strength

$$H = 3\partial B. \quad (3.6)$$

¹⁰This is the parameter m_R of Refs. [21, 23].

These field strengths and the Lagrangian are invariant under the bosonic gauge transformations

$$\delta B = 2\partial\Lambda, \quad \delta C^{(1)} = \partial\Lambda^{(0)} - m\Lambda, \quad \delta C^{(3)} = 3\partial\Lambda^{(2)} - 3mB\Lambda - H\Lambda^{(0)}, \quad (3.7)$$

where the gauge parameters $\Lambda_{\mu\nu}^{(2)}, \Lambda_\mu, \Lambda^{(0)}$ are related to the 11-dimensional ones $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ and $\hat{\xi}^{\hat{\mu}}$ (the generator of infinitesimal g.c.t.'s) by

$$\hat{\chi}_{\mu\nu} = \Lambda_{\mu\nu}^{(2)}, \quad \hat{\chi}_{\mu\bar{z}} = \frac{1}{m}\hat{\lambda}_\mu = \Lambda_\mu, \quad \hat{\xi}^{\bar{z}} = \Lambda^{(0)}. \quad (3.8)$$

The massive gauge invariance of this theory does not lead to a gauged supergravity just because the dimensional reduction of the massive gauge parameter only gives a 1-form. On the other hand, it allows us to gauge away the RR vector leaving in the action a mass term for the NSNS 2-form. This is the simplest example of the Stückelberg mechanism that will be at work in all the cases that we are going to review here. The Stückelberg vectors will always be the ones coming from the metric (here the RR vector). The gauge vector fields (if any) will always come from the 3-form.

The Ansatz for the fermionic fields and supersymmetry parameter is

$$\hat{\epsilon} = e^{-\frac{1}{6}\phi} \epsilon, \quad \hat{\psi}_a = e^{\frac{1}{6}\phi} \left(2\psi_a - \frac{1}{3}\Gamma_a\lambda \right), \quad \hat{\psi}_z = \frac{2i}{3}e^{\frac{1}{6}\phi}\Gamma_{11}\lambda, \quad (3.9)$$

and leads to the supersymmetry transformation rules

$$\begin{cases} \delta_\epsilon \psi_\mu &= \left\{ \partial_\mu - \frac{1}{4} \left(\not{\omega}_\mu + \frac{1}{2}\Gamma_{11} \not{H}_\mu \right) \right\} \epsilon + \frac{i}{8} e^\phi \sum_{n=0}^{n=2} \frac{1}{(2n)!} \mathcal{G}^{(2n)} \Gamma_\mu (-\Gamma_{11})^n \epsilon, \\ \delta_\epsilon \lambda &= \left[\not{\partial}\phi + \frac{1}{2\cdot 3!} \Gamma_{11} \not{H} \right] \epsilon + \frac{i}{4} e^\phi \sum_{n=0}^{n=2} \frac{5-2n}{(2n)!} \mathcal{G}^{(2n)} (-\Gamma_{11})^n \epsilon. \end{cases} \quad (3.10)$$

where we have identified

$$G^{(0)} = m. \quad (3.11)$$

4 Massive $N = 2, d = 9$ Supergravities from $d = 11$

The reduction of the $n = 2$ case in the direction of the two Killing vectors present with the same Kaluza-Klein Ansatz as in the massless case gives gauged/massive $N = 2, d = 9$ supergravities characterized by the mass matrices Q^{mn} [17, 19, 20, 21, 22, 23]. The field content of these theories is

$$\{g_{\mu\nu}, \varphi, L_m^i, C_{\mu\nu\rho}, B_{m\mu\nu}, V_\mu, A^m_\mu, \psi_\mu^i, \lambda^i\}. \quad (4.1)$$

The L_m^i parametrize an $Sl(2, \mathbb{R})/SO(2)$ coset. The field V_μ comes from the 11-dimensional 3-form components $\hat{C}_{\mu mn}$ and will be a gauge field. Its presence is the main new feature with respect to the $n = 1$ case. The gauge group will depend on the choice of mass matrix, as we are going to see. As in all cases, there will always be the same number of 2-forms $B_{\mu\nu m}$ and Kaluza-Klein vectors A^m_μ that play the role of Stückelberg fields for the 2-forms.

Explicitly, the Kaluza-Klein Ansatz for the bosonic fields is

$$(\hat{e}_{\hat{\mu}}^{\hat{a}}) = \begin{pmatrix} e^{-\frac{1}{3\sqrt{7}}\varphi} e_{\mu}^a & e^{\frac{\sqrt{7}}{6}\varphi} L_m^i A^m_{\mu} \\ 0 & e^{\frac{\sqrt{7}}{6}\varphi} L_m^i \end{pmatrix}, \quad (\hat{e}_{\hat{a}}^{\hat{\mu}}) = \begin{pmatrix} e^{\frac{1}{3\sqrt{7}}\varphi} e_a^{\mu} & -e^{\frac{1}{3\sqrt{7}}\varphi} A^m_a \\ 0 & e^{-\frac{\sqrt{7}}{6}\varphi} L_i^m \end{pmatrix}, \quad (4.2)$$

and¹¹

$$\begin{cases} \hat{C}_{\mu\nu\rho} &= C_{\mu\nu\rho} - \frac{3}{2} A^m_{[\mu} B_{m|\nu\rho]} + 3\eta_{mn} V_{[\mu} A^m_{\nu} A^n_{\rho]}, \\ \hat{C}_{\mu\nu m} &= B_{m\mu\nu} - 2\eta_{mn} V_{[\mu} A^n_{\nu]}, \\ \hat{C}_{\mu mn} &= \eta_{mn} V_{\mu}. \end{cases} \quad (4.3)$$

The gauge parameter $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ gives rise to a scalar parameter σ , two vector parameters $\lambda_{m\mu}$ and a 2-form parameter $\chi_{\mu\nu}$:

$$\hat{\chi}_{\mu\nu} = \chi_{\mu\nu}, \quad \hat{\chi}_{\mu m} = \lambda_{m\mu}, \quad \hat{\chi}_{mn} = \eta_{mn}\sigma, \quad (4.4)$$

The gauge vector V_{μ} transforms under the group generated by the single¹² local parameter $\sigma(x)$

$$\delta_{\sigma} V_{\mu} = \partial_{\mu} \sigma. \quad (4.5)$$

To find which is the one-parameter gauge group we have to look at the δ_{σ} transformations of the fields that carry $Sl(2, \mathbb{R})$ indices m, n :

$$\begin{aligned} \delta_{\sigma} L_m^i &= -\sigma L_n^i \mathbf{m}^n_m, \\ \delta_{\sigma} A^m_{\mu} &= \sigma \mathbf{m}^m_n A^n_{\mu}, \\ \delta_{\sigma} B_{\mu\nu m} &= -\sigma B_{m\mu\nu} \mathbf{m}^n_m + 2\eta_{mn} \partial_{[\mu} \sigma A^n_{\nu]}, \end{aligned} \quad (4.6)$$

that leave invariant all the field strengths except for that of $B_{m\mu\nu}$ that transforms covariantly. This tells us that the gauge group of the 9-dimensional theory is the group generated by the 2×2 traceless matrix $\mathbf{m}^m_n = -\mathbf{Q}^{mp} \eta_{pn}$, which is a generator of a subgroup of $Sl(2, \mathbb{R})$. By construction, it is the subgroup that preserves the mass matrix \mathbf{Q}^{mn} : it transforms according to

$$\mathbf{Q}' = \Lambda \mathbf{Q} \Lambda^T, \quad \Lambda = e^{\sigma \mathbf{m}}, \quad (4.7)$$

then the condition that it is preserved $\Lambda^{-1} \mathbf{Q} = \mathbf{Q} \Lambda^T$ translates into

$$\mathbf{m} \mathbf{Q} = -\mathbf{Q} \mathbf{m}^T, \quad (4.8)$$

¹¹The definition of $C_{\mu\nu\rho}$ is not the most naive $\hat{C}_{abc} \sim C_{abc}$ because in this case one is interested in recovering exactly the theories obtained by non-geometrical Scherk-Schwarz reduction from $N = 2B, d = 10$ supergravity [17].

¹²Some of the gauged/massive $N = 2, d = 9$ theories presented in Ref.[23] have a 2-parameter non-Abelian gauge group and, therefore, cannot be described in this framework even if we allowed for more general, non-symmetric mass matrices.

which is trivially satisfied for $\mathbf{m} = -\mathbf{Q}\eta$ on account of the property $\mathbf{m}^T = -\eta\mathbf{m}\eta^{-1}$.

It is clear that the theories obtained can be classified first by the sign of the determinant of the mass matrix $\alpha^2 = -4\det \mathbf{Q}$, which is an $SL(2, \mathbb{R})$ invariant: class I $\alpha^2 = 0$, class II $\alpha^2 > 0$ and class III $\alpha^2 < 0$ [20, 21]. These classes should be subdivided further into $Sl(2, \mathbb{Z})$ equivalence classes since the theories are equivalent only when they are related by $Sl(2, \mathbb{Z})$ transformations. However, it should be clear that theories within the same α^2 class have the same gauge group, the difference being a change of basis which is an $Sl(2, \mathbb{R})$ but not an $Sl(2, \mathbb{Z})$ transformation.

Thus, all theories in class III ($\alpha^2 < 0$) have gauge group $SO(2)$ and all theories in class II ($\alpha^2 > 0$) have gauge group $SO(1, 1)$. The theories in class I ($\alpha^2 = 0$) are all equivalent to one with

$$\mathbf{Q} = \begin{pmatrix} -m & 0 \\ 0 & 0 \end{pmatrix}, \quad (4.9)$$

which is just the reduction of the $n = 1$ case (Romans' theory) considered in the previous section. The group is now $SO(1, 1)$, with

$$\Lambda = \begin{pmatrix} 1 & \sigma m \\ 0 & 1 \end{pmatrix}. \quad (4.10)$$

The transformation laws of the fields of this theory are rather unconventional but the theory is still a gauged supergravity.

From a combination of different terms we get the scalar potential

$$\mathcal{V}(\varphi, \mathcal{M}) = \frac{1}{2} e^{\frac{4}{\sqrt{7}}\varphi} \text{Tr}(\mathbf{m}^2 + \mathbf{m}\mathcal{M}\mathbf{m}^T\mathcal{M}^{-1}), \quad \mathcal{M}_{mn} = L_m^i L_n^i. \quad (4.11)$$

Its presence suggests the existence of domain-wall (7-brane) solutions which will be the vacua of the different theories obtained from different mass matrices. In fact, these domain-wall solutions correspond to different 7-brane solutions of the $N = 2B, d = 10$ theory: each kind of 10-dimensional 7-brane is characterized by its $Sl(2, \mathbb{Z})$ monodromy Λ and it is possible to reduce the $N = 2B, d = 10$ theory in the Scherk-Schwarz generalized fashion admitting this monodromy for the different fields. The result is a gauged/massive $N = 2, d = 9$ supergravity with a mass matrix \mathbf{Q} related to $\Lambda = e^{2\pi R\mathbf{m}}$ as explained above. The domain-wall solutions and their 10-dimensional origin and monodromies have been studied in detail in Ref. [21]. A study of the non-conformal 8-dimensional field theories living in the “boundaries” of these solutions and their relations is still lacking.

The 2-forms $B_{m\mu\nu}$ are also invariant under the standard 2-form gauge transformations

$$\delta_\lambda B_{m\mu\nu} = 2\partial_{[\mu}\lambda_{|m|\nu]}. \quad (4.12)$$

This is possible because these transformations are supplemented by the massive gauge transformations of the KK vectors

$$\delta_\lambda A^m{}_\mu = \mathbf{Q}^{mn}\lambda_{n\mu}, \quad (4.13)$$

that leave invariant the field strength

$$F^m{}_{\mu\nu} = 2\partial_{[\mu}A^m{}_{\nu]} - \mathbf{Q}^{mn}B_{n\mu\nu}, \quad (4.14)$$

which would allow us to gauge them away giving explicit mass terms to the 2-forms. It is in this way (Stückelberg mechanism) that there is no clash between the gauge invariance under δ_σ and the 2-form gauge transformations δ_λ .

The full bosonic action for this theory can be found in Ref. [17] and the fermionic supersymmetry transformation rules in Ref. [18]. Their are not essential to our quick review and we will not write them explicitly.

5 Massive $N = 4, d = 8$ Supergravities from $d = 11$

The reduction of next case $n = 3$ in the direction of the three Killing vectors gives 8-dimensional gauged theories [41]. Only the $SO(3)$ case was studied in Ref. [41] but we are going to show that more general non-compact and non-semisimple gaugings naturally arise as in the previous case. We are going to use the general formalism and field definitions that will be valid in any dimension to show that in the general case n one can get $SO(n-l, l)$ -gauged $(11-n)$ -dimensional supergravities.

The field content of these theories is

$$\{g_{\mu\nu}, \varphi, a, L_m^i, C_{\mu\nu\rho}, B_{m\mu\nu}, V_{mn\mu}, A^m_\mu, \psi_\mu^i, \lambda^i\}, \quad (5.1)$$

where the indices $m, n = 1, 2, 3$ are $Sl(3, \mathbb{R})$ indices and also, simultaneously, gauge indices. The L_m^i parametrize now an $Sl(3, \mathbb{R})/SO(3)$ coset and the three vector fields $V_{mn\mu}$ gauge a 3-dimensional group which should be a subgroup of $Sl(3, \mathbb{R})$ ¹³.

The Anstaz for the bosonic fields is¹⁴

$$(\hat{e}_{\hat{\mu}}^{\hat{a}}) = \begin{pmatrix} e^{\frac{1}{6}\varphi} e_\mu^a & e^{-\frac{1}{3}\varphi} L_m^i A^m_\mu \\ 0 & e^{-\frac{1}{3}\varphi} L_m^i \end{pmatrix}, \quad (\hat{e}_{\hat{a}}^{\hat{\mu}}) = \begin{pmatrix} e^{-\frac{1}{6}\varphi} e_a^\mu & -e^{-\frac{1}{6}\varphi} A^m_a \\ 0 & e^{\frac{1}{3}\varphi} L_i^m \end{pmatrix}, \quad (5.2)$$

and, using now the standard decomposition¹⁵

$$\begin{aligned} \hat{C}_{abc} &= e^{-\frac{1}{2}\varphi} C_{abc}, & \hat{C}_{abi} &= L_i^m B_{mab}, \\ \hat{C}_{aij} &= e^{\frac{1}{2}\varphi} L_i^m L_j^n V_{mna}, & \hat{C}_{ijk} &= e^\varphi \epsilon_{ijk} a, \end{aligned} \quad (5.3)$$

as we are going to do from now on, we get¹⁶

$$\begin{aligned} \hat{C}_{\mu\nu\rho} &= C_{\mu\nu\rho} + 3A^m_{[\mu} B_{m|\nu\rho]} + 3V_{mn[\mu} A^m_\nu A^p_{\rho]} + a\epsilon_{mnp} A^m_{[\mu} A^n_\nu A^p_{\rho]}, \\ \hat{C}_{\mu\nu m} &= B_{m\mu\nu} - 2V_{mn[\mu} A^n_{\nu]} + a\epsilon_{mnp} A^n_\mu A^p_\nu, \\ \hat{C}_{\mu mn} &= V_{mn\mu} + a\epsilon_{mnp} A^p_\mu, \\ \hat{C}_{mnp} &= \epsilon_{mnp} a. \end{aligned} \quad (5.4)$$

¹³In the general case we will have $Sl(n, \mathbb{R})$ indices, the L_m^i will parametrize an $Sl(n, \mathbb{R})/SO(n)$ coset and instead of one scalar a one gets a_{mnp} (here $a_{mnp} = a\epsilon_{mnp}$). Further, we will have $n(n-1)/2$ gauge vectors $V_{mn\mu}$.

¹⁴In the general case only the powers of φ are different.

¹⁵In the general case $\hat{C}_{ijk} \sim L_i^m L_j^n L_k^p a_{mnp}$.

¹⁶In the general case we only have to substitute $a\epsilon_{mnp}$ by a_{mnp} .

The standard decomposition of the 4-form field strength

$$\begin{aligned}\hat{G}_{abcd} &= e^{-\frac{2}{3}\varphi} G_{abcd}, & \hat{G}_{abci} &= e^{-\frac{1}{6}\varphi} L_i^m H_{mabc}, \\ \hat{G}_{abij} &= e^{\frac{1}{3}\varphi} L_i^m L_j^n [F_{mnab} + a\epsilon_{mnp} F^p_{ab}], & \hat{G}_{aijk} &= e^{\frac{5}{6}\varphi} \epsilon_{ijk} \partial_a a,\end{aligned}\tag{5.5}$$

gives the following field strengths

$$\begin{aligned}G_{\mu\nu\rho\sigma} &= 4\partial_{[\mu} C_{\nu\rho\sigma]} + 6B_{m[\mu\nu} F^m_{\rho\sigma]} - 3B_{m[\mu\nu} Q^{mn} B_{n|\rho\sigma]}, \\ H_{m\mu\nu\rho} &= 3\partial_{[\mu} B_{m|\nu\rho]} + 3V_{mn[\mu} F^n_{\nu\rho]}, \\ F_{mn\mu\nu} &= 2\partial_{[\mu} V_{mn|\nu]} + 2V_{mp[\mu} Q^{pq} V_{nq|\nu]}, \\ F^m_{\mu\nu} &= 2\partial_{[\mu} A^m_{\nu]} - Q^{mn} B_{m\mu\nu}.\end{aligned}\tag{5.6}$$

to which we have to add the covariant derivative of the $Sl(3, \mathbb{R})/SO(3)$ scalars¹⁷

$$\mathcal{D}_\mu L_m^i = \partial_\mu L_m^i - V_{mp\mu} Q^{pq} L_q^i,\tag{5.8}$$

Let us now analyze the different gauge symmetries of the theory. The 2-form $\hat{\chi}_{\hat{\mu}\hat{\nu}}$ decomposes now into a 2-form $\chi_{\mu\nu}$, 3 vector parameters $\lambda_{m\mu}$ which will be associated to the massive gauge invariances of the 3 2-forms $B_{m\mu\nu}$ and 3 scalars $\sigma_{mn} = -\sigma_{nm}$ ¹⁸

$$\hat{\chi}_{\mu\nu} = \chi_{\mu\nu}, \quad \hat{\chi}_{\mu m} = \lambda_{m\mu}, \quad \hat{\chi}_{mn} = \sigma_{mn}.\tag{5.9}$$

It is also convenient to define

$$\sigma^m_n = Q^{mp} \sigma_{pn}.\tag{5.10}$$

These are going to be the infinitesimal generators of the gauge transformations. Observe that, depending on the choice of Q^{mp} , σ^m_n can contain an equal or smaller number of independent components than σ_{pn} and, thus, the gauge group can have dimension 3 or smaller.

Under the δ_σ transformations¹⁹

¹⁷In the general case we also have to add the covariant derivative of the a_{mnp} scalars

$$\mathcal{D}_\mu a_{mnp} = \partial_\mu a_{mnp} - 3V_{[m|q\mu} Q^{qr} a_{|np]r},\tag{5.7}$$

that reduces to a partial derivative in $d = 8$ when $a_{mnp} = \epsilon_{mnp} a$.

¹⁸In the general case we will get n vector parameters associated to the massive gauge invariances of the n 2-forms $B_{m\mu\nu}$ and $n(n-1)/2$ scalars $\sigma_{mn} = -\sigma_{nm}$.

¹⁹In the general case the gauge transformations of these fields take the same form but the scalars a_{mnp} transform covariantly

$$\delta_\sigma a_{mnp} = -3a_{q[np} \sigma^q_{m]}.\tag{5.11}$$

This transformation vanishes in $d = 8$ when $a_{mnp} = \epsilon_{mnp} a$.

$$\begin{aligned}
\delta_\sigma L_m^i &= -L_n^i \sigma_m^n, \\
\delta_\sigma A_\mu^m &= \sigma_n^m A_\mu^n, \\
\delta_\sigma V_{mn\mu} &= \mathcal{D}_\mu \sigma_{mn}, \\
\delta_\sigma B_{m\mu\nu} &= -B_{n\mu\nu} \sigma_m^n + 2\partial_{[\mu} \sigma_{mn} A_{\nu]}^n, \\
\delta_\sigma C_{\mu\nu\rho} &= 3\partial_{[\mu} \sigma_{mn} A_{\nu]}^m A_{\rho]}^n,
\end{aligned} \tag{5.12}$$

the field strengths and covariant derivatives transform covariantly, i.e.

$$\begin{aligned}
\delta_\sigma G &= 0, \quad \delta_\sigma H_m = -H_n \sigma_m^n, \quad \delta_\sigma F_{mn} = -2F_{p[n} \sigma_{m]}^p, \quad \delta_\sigma F^m = \sigma_n^m F^n, \\
\delta_\sigma \mathcal{D} L_m^i &= -(\mathcal{D} L_n^i) \sigma_m^n, \quad \delta_\sigma \mathcal{D} a_{mnp} = -3(\mathcal{D} a_{q[np}) \sigma_{m]}^q.
\end{aligned} \tag{5.13}$$

The gauge group is the orthogonal subgroup of $Sl(3, \mathbb{R})$ ($Sl(n, \mathbb{R})$ in the general case) obtained by exponentiation of σ_m^n that preserves the mass matrix \mathbf{Q}^{mn} , i.e.

$$\Lambda \mathbf{Q} \Lambda^T = \mathbf{Q}, \quad \Lambda = e^\sigma. \tag{5.14}$$

It is easy to see that the infinitesimal form of the above condition

$$\sigma_p^m \mathbf{Q}^{pn} = -\sigma_p^n \mathbf{Q}^{mp}, \tag{5.15}$$

is trivially satisfied. The generators of the gauge group in this representation are the matrices

$$\Gamma(M^{mn})^p{}_q = 2\mathbf{Q}^{m[p} \delta^{n]}{}_q, \quad \Rightarrow \sigma_p^p = \frac{1}{2} \sigma_{mn} \Gamma(M^{mn})^p{}_q, \tag{5.16}$$

and the algebra they satisfy can be easily computed

$$[M^{pq}, M^{rs}] = \frac{1}{2} f^{pqrs}{}_{mn} M^{mn}, \quad f^{pqrs}{}_{mn} = 8\delta^{[p}{}_{[m} \mathbf{Q}^{q]}{}_{[n} \delta^{s]}{}_{n]}. \tag{5.17}$$

Actually, using the above structure constants $f^{pqrs}{}_{mn}$ the gauge field strength F_{mn} can be written in the standard form

$$F_{mn} = 2\partial V_{mn} + \frac{1}{4} f^{pqrs}{}_{mn} V_{pq} V_{rs}. \tag{5.18}$$

It is clear that the gauge groups $SO(3)$ and $SO(2, 1)$ correspond to the non-singular diagonal mass matrices²⁰ $\mathbf{Q} = \pm g \text{diag}(+++)$ and $\mathbf{Q} = \pm g \text{diag}(++-)$ respectively, but other groups can also appear. For $n = 3$ we can easily classify all the gauge groups that appear, comparing with the Bianchi classification of all real 3-dimensional Lie algebras²¹.

It is useful to change the notation first:

²⁰It is also clear that in the general case all the orthogonal subgroups $SO(n-l, l)$ of $Sl(n, \mathbb{R})$ are also included.

²¹This study is more complicated for $n > 3$ and, further, the real Lie algebras are not classified in general, but only for $n = 3$ (the well-known Bianchi classification) and $n = 4$. See e.g. Ref. [50] and references therein.

$$V^m{}_\mu \equiv \frac{1}{2}\epsilon^{mnp}V_{np\mu}, \quad T_m \equiv \frac{1}{2}\epsilon_{mnp}M^{np}, \quad f_{uv}{}^w \equiv \frac{1}{8}\epsilon_{upq}\epsilon_{vrs}\epsilon^{wmn}f^{pqrs}{}_{mn} = -\epsilon_{uvw}Q^{tw}. \quad (5.19)$$

The Bianchi classification starts with the observation that the most general structure constants for a 3-dimensional Lie algebra can be written in the form

$$f_{ij}{}^k = -\epsilon_{ijl}b^{lk} + 2\delta_{[i}{}^k a_{j]}, \quad b^{lk} = b^{kl}, \quad b^{kl}a_l = 0. \quad (5.20)$$

The next step consists in the diagonalization of the symmetric matrix b^{kl} whose eigenvalues are normalized to $\pm 1, 0$ and the determination of their possible null eigenvectors a_i . Comparing with the structure constants Eq. (5.19), we see that we can obtain all the Lie algebras in the Bianchi classification with $a_i = -\frac{1}{2}f_{ik}{}^k = 0$. These are the Bianchi I, II, VI, VII, VIII ($so(2,1)$) and IX ($so(3)$) algebras.

How about the remaining Bianchi III, IV and V algebras? It is a simple exercise to rewrite the general structure constants Eq. (5.20) in terms of just one constrained matrix $d^{lk} = b^{kl} - \epsilon^{kli}a_i$ with no definite symmetry properties:

$$f_{ij}{}^k = \epsilon_{ijl}d^{lk}, \quad d^{(lk)}\epsilon_{kij}d^{ij} = 0. \quad (5.21)$$

This seems to suggest that we could cover all the possible cases by allowing for a general, non-symmetric Q^{mn} , but this has to be checked in detail.

Let us conclude the study of the gauge symmetries of the theory: the parameters $\lambda_{m\mu}$ generate massive gauge transformations under which

$$\delta_\lambda A^m{}_\mu = Q^{mn}\lambda_{n\mu}, \quad \delta_\lambda B_{m\mu\nu} = 2\partial_{[\mu}\lambda_{m|\nu]}, \quad \delta_\lambda C_{\mu\nu\rho} = -6A^m{}_{[\mu}\partial_{\nu]}\lambda_{m|\rho]}, \quad (5.22)$$

leaving invariant all the field strengths. In this and all cases this invariance can be used to eliminate the 3 (n) KK vector fields $A^m{}_\mu$ giving masses to the 3 (n) 2-forms $B_{m\mu\nu}$. The action for the full theory can be found in Ref. [41].

Let us now compare the theory obtained, with $Q = g\mathbb{I}_{3\times 3}$ and gauge group $SO(3)$ (the other cases cannot be compared) with Salam & Sezgin's (SS) [12]. The field contents are identical, only the couplings are different: in the SS case the gauge vector fields are the KK ones $A^m{}_\mu$ and the Stückelberg vector fields are the ones coming from the 3-form $V^m{}_\mu$, while in our case these roles are interchanged (the 2-forms are always massive). Some of the couplings to the scalars φ and a are also different.

Actually it is convenient to describe the differences between both 8-dimensional theories through the action of the global $Sl(2, \mathbb{R})$ duality symmetry that the (equations of motion of the) massless theory enjoys. The scalars φ and a can be combined in the complex scalar $\tau = a + ie^{-\varphi}$ that parametrizes the coset $Sl(2, \mathbb{R})/SO(2)$ and undergoes fractional-linear transformations under $Sl(2, \mathbb{R})$. Under this group, the vector fields form 3 doublets $(V^m{}_\mu, A^m{}_\mu)$ while the 2-forms are singlets²². The 4-form field strength G undergoes electric-magnetic duality rotations.

The differences between the two 8-dimensional gauged theories are associated, precisely, to the $Sl(2, \mathbb{R})$ transformation

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (5.23)$$

²²Actually, the 2-forms are singlets after a field redefinition.

that interchanges the vector fields V^m_μ and A^m_μ and transforms τ into $-1/\tau$. This is reflected in the scalar potential which in our case is given by

$$\mathcal{V} = -\frac{1}{2} \frac{|\tau|^2}{\Im(\tau)} \{ [\text{Tr}(\mathcal{QM})]^2 - 2\text{Tr}(\mathcal{QM})^2 \} , \quad (5.24)$$

while in SS's case is

$$\mathcal{V}_{\text{SS}} = -\frac{1}{2} g^2 \frac{1}{\Im(\tau)} \{ (\text{Tr} \mathcal{M})^2 - 2\text{Tr}(\mathcal{M}^2) \} . \quad (5.25)$$

Thus we can view our theory as the S-dual of SS's although, in practice, one cannot perform such a transformation directly on the gauged theories and, rather, one would have to do it in the ungauged one.

The non-compact gaugings that we obtain from “massive 11-dimensional supergravity” have no known “S-dual”, although it should be possible to obtain them by the analytical continuation methods of Ref. [39]. Their 11-dimensional origin is unknown. As for the non-semisimple gaugings, their “S-duals” are also unknown, but now analytical continuation cannot be used to construct them.

6 Massive $N = 8, d = 5$ Supergravities from $d = 11$

From the discussions and examples in the previous sections it should be clear that in the $n = 4$ case we will obtain $SO(4-l, l)$ -gauged 7-dimensional supergravities etc. A particularly interesting case is the $n = 6$ one, in which we can obtain $SO(6-l, l)$ -gauged $N = 8, d = 5$ supergravities which were constructed in Refs. [28, 29]. This offers us the possibility to check our construction and show that, as we have claimed, it systematically gives gauged/massive supergravities.

The derivation of the 5-dimensional theory from “massive 11-dimensional supergravity” offers no new technical problems and the action, field strengths etc. can be found applying the general recipes explained in the previous section and are written explicitly in Appendix A. One of the highlights of this derivation is the field content which is of the general form

$$\{g_{\mu\nu}, \varphi, a, a_{mnp}, L_m^i, B_{m\mu\nu}, V_{mn\mu}, A^m_\mu, \psi_\mu^i, \lambda^i\} , \quad (6.1)$$

where now the m, n, p indices are $Sl(6, \mathbb{R})$ indices and where we have dualized the 3-form $C_{\mu\nu\rho}$ into the scalar a . The scalars φ and a can be combined again into the complex τ that parametrizes $Sl(2, \mathbb{R})/SO(2)$. In the ungauged/massless theory this $Sl(2, \mathbb{R})$ global symmetry and the more evident $Sl(6, \mathbb{R})$ are part of the E_6 duality group of the theory that only becomes manifest after the 6 2-forms are also dualized into 6 additional vector fields²³.

As usual, this is also the field content of the ungauged theory. This is already a surprise since in Refs. [28, 29] it was argued that the theory could only be consistently gauged if the 6 KK vector fields A^m_μ were dualized into 6 2-forms $\tilde{B}_{m\mu\nu}$ which, together with the already existing 6 2-forms $B_{m\mu\nu}$ and via a self-dual construction, could describe 6 massive 2-forms. Once there are no massless higher-rank fields with $Sl(6, \mathbb{R})$ indices left, the theory

²³The bosonic action of the massless theory with $C_{\mu\nu\rho}$ dualized into a and the $B_{m\mu\nu}$ dualized into vector fields N^m_μ is given in Eq. (A.9).

can be consistently gauged. In the theory that we get, the same goal is achieved by the Stückelberg mechanism: the 6 KK vector fields A^m_μ are not dualized but are gauged away leaving mass terms for the already existing 6 2-forms $B_{m\mu\nu}$.

Another interesting point is the form of the scalar potential $\mathcal{V}(\varphi, a_{mnp})$, given in Eq. (A.14). The first term, which is universal for all the gauged/massive theories we are studying and is the only one that survives the consistent truncation $a_{mnp} = 0$, is independent of the scalar φ that measures the volume of the internal manifold. As shown in Appendix B, this universal term is always minimized for $\mathcal{M} = \mathbb{I}_{n \times n}$ when $\mathbf{Q} = g \mathbb{I}_{n \times n}$ and the value of the potential for $n = 6$ is constant and the vacuum is AdS_5 as in Refs. [28, 29]. Not only the vacuum is the same: in Ref. [43] it was shown that there is a consistent truncation of the scalars that leaves the same potential (the first term of Eq. (A.14)) for the remaining scalars and thus, in spite of the apparent differences it is plausible that the two untruncated potentials are completely equivalent.

If the field content is equivalent, the symmetries of the theory are the same the vacuum is the same and, presumably the potentials are equivalent, we can expect to have obtained a completely equivalent form of the $SO(6-l, l)$ -gauged $N = 8, d = 5$ theories constructed in Refs. [28, 29]. To show or, rather, to make more plausible this equivalence we would like to show that these theories have identical equations of motion, but this is extremely complicated for the full theories and we will content ourselves with showing the equivalence of the self-dual and Stückelberg Lagrangians for charged 2-forms ignoring all the scalars for the sake of simplicity.

6.1 Self-Duality versus Stückelberg

The gauging of $N = 4, d = 7$ [38] and $N = 8, d = 5$ [28, 29] supergravity theories presents many peculiar features and problems absent in other cases. All these problems were resolved using the *self-duality mechanism* [34, 33]. Before comparing it with the Stückelberg mechanism, we will review the above mentioned problems and the reasoning that lead to the use of the self-duality mechanism to solve them in the 5-dimensional case.

In the usual gauging procedure one gauges the symmetry group of all the vector fields present in the ungauged theory. In one version of $N = 8, d = 5$ ungauged supergravity in which all 2-forms have been dualized into vectors, there are 27 vector fields, but there is no 27-dimensional simple Lie group, and therefore the standard recipe does not work. The origin of the gauged theory from IIB supergravity compactified on S^5 [9, 10, 44] suggested the gauging of the isometry group of the internal space, the 15-dimensional $SO(6)$. $E_{6(6)}$ being the global symmetry group of the ungauged theory and $Usp(8)$ the local composite one, the idea was to gauge an $SO(6)$ subgroup of the $Usp(8)$ embedded in $E_{6(6)}$ ²⁴. All bosonic fields are in irreducible representations of $E_{6(6)}$ and in general transform as reducible representations under $SO(6)$. In particular, the **27** of vector fields breaks, under $SO(6)$, as $\mathbf{27} = \mathbf{15} + \mathbf{6} + \mathbf{6}$. The **15** is precisely the adjoint of $SO(6)$. This raises a second problem: how to couple the two sextets of abelian vector fields to the 15 $SO(6)$ Yang-Mills fields.

On the other hand, the superalgebra of the gauged theory was expected to be $SU(2, 2|4)$. The irreducible representation of this superalgebra in which the graviton is contained also contains two sextets of 2-index antisymmetric tensor fields (2-forms). This and other

²⁴ $Usp(8)$ contains $Sl(2, \mathbb{R}) \times Sl(6, \mathbb{R})$ as a subgroup, and the $SO(6)$ to be gauged is in $Sl(6, \mathbb{R})$. One may also gauge a non-compact group $SO(6-l, l)$ instead of $SO(6)$.

reasons [9, 10] suggested the replacement of the two sextets of abelian vector fields by two sextets of 2-form fields, but there is also a problem in coupling these fields to the Yang-Mills ones: it is not possible to reconcile both gauge invariances simultaneously. Replacing ordinary derivatives by Yang-Mills covariant ones breaks the local gauge invariance of the antisymmetric fields, which means that there are more modes propagating than in the ungauged theory. But there is a way out: the antisymmetric fields must satisfy self-dual equations of motion (to be described later).

Once the twelve vectors have been replaced by the self-dual 2-form fields one finds that the latter do not satisfy Bianchi identities, and for consistency the model must be gauged [28, 29]. This, in turn, implies that, naively, the gauged theory does not have a good $g \rightarrow 0$ limit, although the limit can be taken after elimination of one of the 2-form sextets [34]. In our (Stückelberg) formulation, the $g \rightarrow 0$ limit can always be taken.

In the next two subsections we are going to construct Stückelberg formulations for a massive, uncharged 2-form field and for a sextet of massive 2-form fields charged under $SO(6)$ Yang-Mills fields and we will show that they lead to equations of motion fully equivalent to those obtained from self-dual formulations. The Stückelberg formulations are just simplifications of our gauged/massive $N = 8, d = 5$ supergravity theory.

6.1.1 Uncharged case

We start from the standard action for a massive 2-form field

$$S[B] = \int d^5x \left\{ \frac{1}{2 \cdot 3!} H^2 - \frac{1}{4} m^2 B^2 \right\} , \quad (6.2)$$

where $H = 3\partial B$. The equation of motion for B derived from (6.2) is the Proca equation

$$(\square + m^2)B_{\mu\nu} = 0 . \quad (6.3)$$

The action given in (6.2) is not gauge invariant. To recover formally gauge invariance we introduce in the action a Stückelberg field A_μ , such that the action is now

$$S[B, A] = \int d^5x \left\{ \frac{1}{2 \cdot 3!} H^2 - \frac{1}{4} F^2 \right\} , \quad (6.4)$$

where

$$H = 3\partial B , \quad (6.5)$$

$$F = 2\partial A - mB .$$

The equations of motion for these fields are

$$\partial_\mu H^{\mu\nu\rho} - mF^{\nu\rho} = 0 , \quad (6.6)$$

$$\partial_\mu F^{\mu\nu} = 0 ,$$

and now we have invariance under the following “massive gauge transformations”:

$$\delta A = m\Lambda , \quad (6.7)$$

$$\delta B = 2\partial\Lambda .$$

The vector A_μ does not propagate any degrees of freedom, since it can be completely gauged away. In fact, setting $A_\mu = 0$ we recover the Proca equation. So, as we know, the introduction of the Stückelberg field is just a way of re-writing the theory described by (6.2) in a formally gauge invariant way.

Now, to connect with the self-dual formulation, we dualize the vector A_μ into a two-form $\tilde{B}_{\mu\nu}$: we add a Lagrange multiplier term in the action:

$$S[B, \tilde{B}, F] = \int d^5x \left\{ \frac{1}{2 \cdot 3!} H^2 - \frac{1}{4} m^2 B^2 + \frac{1}{4} \epsilon \partial \tilde{B} (F + mB) \right\}. \quad (6.8)$$

The equation of motion for $F = dA$ is

$$F = {}^* \tilde{H} = \frac{1}{3!} \epsilon \tilde{H}, \quad (6.9)$$

where $\tilde{H} = 3\partial\tilde{B}$. Inserting Eq. (6.9) into (6.8) one gets

$$S[B, \tilde{B}] = \int d^5x \left\{ \frac{1}{2 \cdot 3!} H^2 + \frac{1}{2 \cdot 3!} \tilde{H}^2 + \frac{m}{12} \epsilon \tilde{H} B \right\}. \quad (6.10)$$

The action above contains two 2-forms, but it describes the degrees of freedom of only one massive 2-form. Observe that this action is invariant under the gauge transformations

$$\delta B = 2\partial\Sigma, \quad \delta \tilde{B} = 2\partial\tilde{\Sigma}. \quad (6.11)$$

Using this gauge invariance, the equations of motion derived from (6.10) can always be integrated to yield²⁵

$$\begin{aligned} {}^* H &= +m\tilde{B}, \\ {}^* \tilde{H} &= -mB, \end{aligned} \quad (6.12)$$

which are precisely the equations of motion that one can derive directly from the *self-dual* action:

$$S_{SD}[B, \tilde{B}] = \int d^5x \left\{ -\frac{1}{4} m^2 B^2 - \frac{1}{4} m^2 \tilde{B}^2 - \frac{m}{12} \epsilon \tilde{H} B \right\}. \quad (6.13)$$

Therefore, the self-dual action Eq. (6.13) and (6.10) (and, therefore, the Stückelberg action Eq. (6.4)) are classically equivalent actions since they lead to the same equations of motion.

Our next step will be to establish a relation between the Stückelberg and self-dual actions for a sextet of $SO(6)$ -charged, massive 2-forms.

6.1.2 The $SO(6)$ Charged Case

Let us consider now six massive two forms coupled to the 15 $SO(6)$ -vector fields V_{mn} . The Stückelberg action for them can be read off from the action describing the 5-dimensional massive/gauged supergravity, given explicitly in Appendix (A) setting $Q^{mn} = m\delta^{mn}$. To simplify matters we truncate all the fields that are not relevant for our problem (in particular, all the scalars) and will work in flat spacetime. We are left with

²⁵These two equations can be combined to get the Proca equation.

$$S[B_m, A_m, V_{mn}] = \int d^5x \left\{ \frac{1}{2 \cdot 3!} \mathbb{H}_m \mathbb{H}_m - \frac{1}{4} F_m F_m - \frac{1}{4} \mathcal{F}_m \mathcal{F}_m \right\}, \quad (6.14)$$

where

$$\begin{aligned} \mathbb{H}_m &= 3\partial B_m + 3V_{mn}F_n \equiv H_m + 3V_{mn}F_n, \\ F_m &= 2\partial A_m - mB_m, \\ \mathcal{F}_{mn} &= 2\partial V_{mn} + 2mV_{mp}V_{np}. \end{aligned} \quad (6.15)$$

where \mathcal{D} is the $SO(6)$ covariant derivative.

This action is invariant under

$$\begin{aligned} \delta A_m &= \sigma_{mn}A_n + m\lambda_m, \\ \delta V_{mn} &= \mathcal{D}\sigma_{mn}, \\ \delta B_m &= 2\partial\lambda_m + 2\partial\sigma_{mn}A_n + m\sigma_{mn}B_n, \end{aligned} \quad (6.16)$$

In order to dualize the vectors A_m into two-forms \tilde{B}_m we follow exactly the same steps as in the uncharged case, and the (much more complicated) equation we find for F_m is

$$F_m{}^{\mu\nu} = \mathcal{P}^{-1}(V)_{mn}{}^{\mu\nu}{}_{\rho\sigma} \left[(*\tilde{H})_n{}^{\rho\sigma} + H_p{}^{\rho\sigma\lambda} V_{pn\lambda} \right], \quad (6.17)$$

where

$$\mathcal{P}_{mn}{}^{\rho\sigma}{}_{\mu\nu}(V) = \delta_{mn} \eta^{[\rho\sigma]}{}_{\mu\nu} - 3\eta^{[\rho\sigma}{}_{\mu\nu} V_{np}{}^{\lambda]} V_{mp\lambda}, \quad (6.18)$$

Then, the action in terms of the dual fields \tilde{B}_m reads

$$\begin{aligned} S[B_m, \tilde{B}_m, V_{mn}] &= \int d^5x \left\{ \frac{1}{2 \cdot 3!} H_m H_m + \frac{1}{4} (*\tilde{H}_m + H_p V_{pm}) \mathcal{P}_{mn}^{-1} (*\tilde{H}_n + H_q V_{qn}) \right. \\ &\quad \left. - \frac{1}{4} \mathcal{F}_{mn} \mathcal{F}_{mn} + \frac{1}{12} \epsilon \tilde{H}_m B_m \right\}. \end{aligned} \quad (6.19)$$

The action given in (6.19) describes only the degrees of freedom of the six massive 2-forms B_m coupled to the vector fields V_{mn} . This action is invariant under the following gauge transformations

$$\begin{aligned} \delta V_{mn} &= \mathcal{D}\sigma_{mn}, \\ \delta B_m &= \mathcal{P}_{mn}^{-1} \left\{ \left(d\lambda_n - \frac{1}{2} \epsilon d\tilde{\lambda}_p V_{np} \right) - \frac{1}{2} \epsilon \left[\left(\tilde{B}_p - \frac{1}{2} \epsilon B_q V_{pq} \right) \mathcal{D}\sigma_{np} \right] \right\}, \\ \delta \tilde{B}_m &= \mathcal{P}_{mn}^{-1} \left\{ \left(d\tilde{\lambda}_n - \frac{1}{2} \epsilon d\lambda_p V_{np} \right) - \frac{1}{2} \epsilon \left[\left(B_p - \frac{1}{2} \epsilon \tilde{B}_q V_{pq} \right) \mathcal{D}\sigma_{np} \right] \right\}. \end{aligned} \quad (6.20)$$

The equations of motion derived from (6.19) are

$$\begin{aligned}
\mathcal{D}_\mu \mathcal{F}_{mn}{}^{\mu\nu} &= \frac{1}{4} m^2 \epsilon^{\nu\rho\sigma\delta\lambda} B_{[m|\rho\sigma} \tilde{B}_{|n]\delta\lambda}, \\
{}^* H_m &= +m[\tilde{B}_m + \frac{1}{2} \epsilon V_{mn} B_n], \\
{}^* \tilde{H}_m &= -m[B_m + \frac{1}{2} \epsilon V_{mn} \tilde{B}_n],
\end{aligned} \tag{6.21}$$

which can also be derived from the *self-dual* action:

$$S_{SD}[\vec{B}_m, V_{mn}] = \int d^5x \left\{ -\frac{1}{4} m^2 \vec{B}_m^T \vec{B}_m - \frac{1}{4} \mathcal{F}_{mn} \mathcal{F}_{mn} - \frac{m}{8} \epsilon \vec{B}_m^T \eta \mathcal{D} \vec{B}_m \right\}, \tag{6.22}$$

where

$$\vec{B}_m = \begin{pmatrix} B_m \\ \tilde{B}_m \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.23}$$

and \mathcal{D} is the $SO(6)$ covariant derivative acting on \vec{B}_m :

$$\mathcal{D} \vec{B}_m = \begin{pmatrix} \partial B_m - m V_{mn} B_n \\ \partial \tilde{B}_m + m V_{mn} \tilde{B}_n \end{pmatrix}. \tag{6.24}$$

Observe that the $SO(6)$ charges of B_m and \tilde{B}_m are opposite.

This is precisely the kind of action that appears in the standard form of $N = 8, d = 5$ gauged supergravity.

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A The Reduction of Massive 11-Dimensional SUGRA to $d = 5$

A.1 Direct dimensional reduction of $D = 11$ Supergravity on T^6

The KK Ansatz for the Elfbein is

$$(\hat{e}_{\hat{\mu}}^{\hat{a}}) = \begin{pmatrix} e^{\frac{1}{3}\varphi} e_{\mu}^a & e^{-\frac{1}{6}\varphi} L_m^i A^m_{\mu} \\ 0 & e^{-\frac{1}{6}\varphi} L_m^i \end{pmatrix}, \quad (\hat{e}_{\hat{a}}^{\hat{\mu}}) = \begin{pmatrix} e^{-\frac{1}{3}\varphi} e_a^{\mu} & -e^{-\frac{1}{3}\varphi} A^m_a \\ 0 & e^{\frac{1}{6}\varphi} L_i^m \end{pmatrix}, \tag{A.1}$$

and for the 3-form potential

$$\begin{aligned}
\hat{C}_{abc} &= e^{-\varphi} C_{abc}, & \hat{C}_{abi} &= e^{-\varphi/2} L_i^m B_{mab}, \\
\hat{C}_{aij} &= L_i^m L_j^n V_{mna}, & \hat{C}_{ijk} &= e^{\varphi/2} L_i^m L_j^n L_k^p \partial_a a_{mnp},
\end{aligned} \tag{A.2}$$

which, in curved components, are given in (5.4).

The 11-dimensional 4-form field strength decomposes as

$$\begin{aligned}
\hat{G}_{abcd} &= e^{-\frac{4}{3}\varphi} G_{abcd}, & \hat{G}_{abci} &= e^{-\frac{5}{6}\varphi} L_i^m H_{mabc}, \\
\hat{G}_{abij} &= e^{-\frac{1}{3}\varphi} L_i^m L_j^n [F_{mnab} + a\epsilon_{mnp} F^p_{ab}], & \hat{G}_{aijk} &= e^{\frac{1}{6}\varphi} L_i^m L_j^n L_k^p \partial_a a_{mnp},
\end{aligned} \tag{A.3}$$

such that the field strengths are

$$\begin{aligned}
G_{\mu\nu\rho\sigma} &= 4\partial_{[\mu} C_{\nu\rho\sigma]} + 6B_{m[\mu\nu} F^m_{\rho\sigma]}, \\
H_{m\mu\nu\rho} &= 3\partial_{[\mu} B_{m|\nu\rho]} + 3V_{mn[\mu} F^n_{\nu\rho]}, \\
F_{mn\mu\nu} &= 2\partial_{[\mu} V_{mn|\nu]}, \\
F^m_{\mu\nu} &= 2\partial_{[\mu} A^m_{\nu]}.
\end{aligned} \tag{A.4}$$

The action for the massless/ungauged 5-dimensional action is

$$\begin{aligned}
S &= \int d^5x \sqrt{|g|} \left\{ R + \frac{1}{2} (\partial\varphi)^2 + \frac{1}{4} \text{Tr} (\partial\mathcal{M}\mathcal{M}^{-1})^2 \right. \\
&\quad - \frac{1}{4} e^{-\varphi} F^m(A) \mathcal{M}_{mn} F^n(A) - \frac{1}{2 \cdot 4!} e^{-2\varphi} G^2 + \frac{1}{2 \cdot 3!} e^{-\varphi} H_m \mathcal{M}^{mn} H_n \\
&\quad - \frac{1}{8} \mathcal{M}^{mp} \mathcal{F}_{mn} \mathcal{M}^{nq} \mathcal{F}_{pq} + \frac{1}{18} e^\varphi \mathcal{M}^{mq} \mathcal{M}^{nr} \mathcal{M}^{ps} \partial a_{mnp} \partial a_{qrs} \\
&\quad - \frac{1}{2^6 \cdot 3^4} \frac{\epsilon}{\sqrt{|g_E|}} \epsilon^{mnpqrs} [2G \partial a_{mnp} a_{qrs} + 12 H_m \mathcal{F}_{np} a_{qrs} \\
&\quad + 24 H_m \partial a_{npq} V_{rs} + 27 \mathcal{F}_{mn} \mathcal{F}_{pq} V_{rs} + 36 \mathcal{F}_{mn} \partial a_{pqr} B_s \\
&\quad \left. + 4 \partial a_{mnp} \partial a_{qrs} C] \right\}.
\end{aligned} \tag{A.5}$$

where

$$\mathcal{F}_{mn} = F_{mn}(V) + a_{mnp} F^p(A). \tag{A.6}$$

In d dimensions the Hodge-dual of a p -form potential is a $d - p - 2$ potential. We are interested in dualizing the three-form and two-form fields C and B_m into scalar and vector potentials. We get

$$\begin{aligned}
G &= e^{2\varphi} * \tilde{G}, \\
H_m &= e^\varphi * \tilde{H}_m,
\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}\tilde{G} &= \partial a - \frac{1}{6^3} \epsilon^{mnpqrs} \partial a_{mnp} a_{qrs} , \\ \tilde{H}_m &= \mathcal{M}_{mn} \left[2\partial N^n - \frac{1}{36} \epsilon^{npqrsu} \mathcal{F}_{pq} a_{rsu} + a F^n(A) \right] .\end{aligned}\tag{A.8}$$

Then, the action (A.5) in terms of the dual fields reads

$$\begin{aligned}S &= \int d^5x \sqrt{|g|} \left\{ R + \frac{1}{4} \text{Tr} (\partial \mathcal{M} \mathcal{M}^{-1})^2 + \frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} e^{2\varphi} \tilde{G}^2 \right. \\ &\quad - \frac{1}{4} e^{-\varphi} F^m(A) \mathcal{M}_{mn} F^n(A) - \frac{1}{4} e^\varphi \tilde{H}_m \mathcal{M}^{mn} \tilde{H}_n \\ &\quad - \frac{1}{8} \mathcal{M}^{mp} \mathcal{F}_{mn} \mathcal{M}^{nq} \mathcal{F}_{pq} + \frac{1}{18} e^\varphi \mathcal{M}^{mq} \mathcal{M}^{nr} \mathcal{M}^{ps} \partial a_{mnp} \partial a_{qrs} \\ &\quad - \frac{1}{2^6 \cdot 3^4} \frac{\epsilon}{\sqrt{|g_E|}} \left[\epsilon^{mnpqrs} (27 \mathcal{F}_{mn} \mathcal{F}_{pq} V_{rs} - 12 a_{mnp} F_{qr}(V) F^u(A) V_{us}) \right. \\ &\quad \left. \left. - 6^4 F^m(A) F_{mn}(V) N^n \right] \right\} .\end{aligned}\tag{A.9}$$

A.2 Dimensional Reduction of Massive 11-Dimensional Supergravity

The decomposition of the 11-dimensional 3-form potential and 4-form field strength are given in (A.2). The 11-dimensional field strength decomposes as in (A.3), but now

$$\hat{G}_{aijk} = e^{\frac{1}{6}\varphi} L_i^m L_j^n L_k^p \mathcal{D}_a a_{mnp} ,\tag{A.10}$$

this is, we have replaced ∂ by the $SO(6-l, l)$ covariant derivative \mathcal{D} . There is also a new contribution from the eleven-dimensional field-strength, namely

$$\hat{G}_{ijkl} = e^{\frac{1}{6}\varphi} L_i^m L_j^n L_k^p L_l^q \left[-3 Q^{rs} a_{r[mn} a_{pq]s} \right] ,\tag{A.11}$$

which will contribute to the scalar potential. The five-dimensional field strengths are now massive, and are defined as in (5.6). The expressions for the massive gauge transformations are the same as those in (5.12) and (5.22).

Then, the $d = 5$ massive action reads

$$\begin{aligned}
S = & \int d^5x \sqrt{|g_E|} \left\{ R_E + \frac{1}{2} (\partial\varphi)^2 + \frac{1}{4} \text{Tr} (\mathcal{D}\mathcal{M}\mathcal{M}^{-1})^2 \right. \\
& - \frac{1}{4} e^{-\varphi} F^m(A) \mathcal{M}_{mn} F^n(A) - \frac{1}{2 \cdot 4!} e^{-2\varphi} G^2 + \frac{1}{2 \cdot 3!} e^{-\varphi} H_m \mathcal{M}^{mn} H_n \\
& - \frac{1}{8} \mathcal{M}^{mp} \mathcal{F}_{mn} \mathcal{M}^{nq} \mathcal{F}_{pq} + \frac{1}{18} e^\varphi \mathcal{M}^{mq} \mathcal{M}^{nr} \mathcal{M}^{ps} \mathcal{D}a_{mnp} \mathcal{D}a_{qrs} - \mathcal{V} \\
& - \frac{1}{2^6 \cdot 3^4} \frac{\epsilon}{\sqrt{|g_E|}} \epsilon^{mnpqrs} \left\{ 2 G \mathcal{D}a_{mnp} a_{qrs} + 12 H_m \mathcal{F}_{np} a_{qrs} + 24 H_m \mathcal{D}a_{npq} V_{rs} \right. \\
& + 27 \mathcal{F}_{mn} \mathcal{F}_{pq} V_{rs} + 36 \mathcal{F}_{mn} \mathcal{D}a_{pqr} B_s + 4 \mathcal{D}a_{mnp} \mathcal{D}a_{qrs} C \\
& + 9 Q^{vw} [2 (GV_{mn} + 4H_m B_n + 2\mathcal{F}_{mn} C) a_{pqv} a_{rsw} \\
& + 2 (Ga_{mnp} + 12H_m V_{np} + 18\mathcal{F}_{mn} B_p + 3\mathcal{D}a_{mnp} C) V_{qv} a_{rsw} \\
& + (4H_m a_{npq} + 18\mathcal{F}_{mn} V_{pq} + 9\mathcal{D}a_{mnp} B_q) B_v a_{rsw} \\
& + 4 (2H_m a_{npq} + 9\mathcal{F}_{mn} V_{pq} + 6\mathcal{D}a_{mnp} B_q) V_{rv} V_{sw} \\
& + 12 (\mathcal{F}_{mn} a_{pqr} + 2\mathcal{D}a_{mnp} V_{qr}) B_v V_{sw} + 3\mathcal{D}a_{mnp} a_{qrs} B_v B_w] \\
& + \frac{9}{10} Q^{vw} Q^{xy} [9 (4a_{mnp} a_{qrv} V_{sw} + 3V_{mn} a_{pqv} a_{rsw}) B_x B_y \\
& + 8 (9V_{mn} a_{pqv} V_{rw} + 16B_m a_{npv} a_{qrw}) V_{sx} B_y \\
& + 24 (3V_{mn} V_{pv} V_{qw} + 6B_m a_{npv} V_{qw} + 12C a_{mnv} a_{pqw}) V_{rx} V_{sy} \\
& \left. \left. + 36C a_{mnv} a_{pqw} a_{rsx} B_y \right] \right\} \} ,
\end{aligned} \tag{A.12}$$

where

$$\mathcal{F}_{mn} = F_{mn}(V) + a_{mnp} F^p(A) \tag{A.13}$$

with $F_{mn}(V)$ and $F^m(A)$ are the massive ones. \mathcal{V} is the scalar potential, given by

$$\begin{aligned}
\mathcal{V} = & -\frac{1}{2} \left\{ [\text{Tr}(\mathcal{M}\mathcal{Q})]^2 - 2 \text{Tr}(\mathcal{M}\mathcal{Q}\mathcal{M}\mathcal{Q}) \right. \\
& - \frac{1}{2} e^\varphi [(\mathcal{Q}\mathcal{M}\mathcal{Q})^{mq} \mathcal{M}^{nr} \mathcal{M}^{ps} - 2 \mathcal{Q}^{mq} \mathcal{M}^{nr} \mathcal{Q}^{ps}] a_{mnp} a_{qrs} \\
& \left. - \frac{1}{24} e^{2\varphi} \mathcal{M}^{mr} \mathcal{M}^{ns} \mathcal{M}^{pt} \mathcal{M}^{qu} (3 a_{v[mn} a_{p]qw} \mathcal{Q}^{vw}) (3 a_{x[rs} a_{t]uy} \mathcal{Q}^{xy}) \right\} .
\end{aligned} \tag{A.14}$$

B Extremizing the Scalar Potential

The above potential is a complicated function on many independent scalar variables plus the parameters of the mass matrix \mathcal{Q} , which makes extremely difficult a complete study of its extrema. Only some extrema are known in the $SO(6-l, l)$ cases [47, 48, 49].

In the general case of the series of gauged/masive supergravities that we can generate from $d = 11$, it is even more difficult to find and study all the extrema of the potential. However, all the potentials contain the “universal term” Eq. (1.1) that only depends on \mathcal{M} and the mass matrix \mathbf{Q} and we can try to systematically study it.

The universal term is a function of the dilaton φ and the matrix of scalars \mathcal{M}

$$\mathcal{V} = \mathcal{V}(\varphi, \mathcal{M}), \quad (\text{B.1})$$

whose origin (in the family of gauged/massive supergravities that we are studying) is the internal metric G_{mn} , which we have decomposed as

$$G_{mn} = e^{-\frac{2}{n}\varphi} \mathcal{M}_{mn}, \quad (\text{B.2})$$

n being the dimension of the compact space, the n -torus.

In the reduced theory the $Sl(n, \mathbb{R})/SO(n)$ matrix of scalars \mathcal{M}_{mn} is subject to the constraint $\det(\mathcal{M}) = 1$, and this has to be taken into account to find its equations of motion. the simplest way to do it is to calculate the equations of motion for the unconstrained variables G_{mn} , and not \mathcal{M}_{mn} and φ separately, and then use the chain rule:

$$\frac{\delta S}{\delta G_{mn}} = 0 \Rightarrow \frac{n}{2} \mathcal{M}^{mn} \frac{\delta S}{\delta \varphi} - \frac{\delta S}{\delta \mathcal{M}_{mn}} = 0. \quad (\text{B.3})$$

The potential \mathcal{V} has to be extremized w.r.t. G_{mn} as well:

$$\frac{\partial \mathcal{V}}{\partial G_{mn}} = 0, \Rightarrow \frac{n}{2} \mathcal{M}^{mn} \frac{\partial \mathcal{V}}{\partial \varphi} - \frac{\partial \mathcal{V}}{\partial \mathcal{M}_{mn}} = 0. \quad (\text{B.4})$$

Contracting the equation above with \mathcal{M}_{mn} we find

$$\frac{\partial \mathcal{V}}{\partial \varphi} = \frac{2}{n} \frac{1}{\text{Tr}(\mathcal{M}^2)} \mathcal{M}_{mn} \frac{\partial \mathcal{V}}{\partial \mathcal{M}_{mn}}, \Rightarrow \mathcal{M}^{mn} \mathcal{M}_{pq} \frac{\partial \mathcal{V}}{\partial \mathcal{M}_{pq}} - \text{Tr}(\mathcal{M}^2) \frac{\partial \mathcal{V}}{\partial \mathcal{M}_{mn}} = 0. \quad (\text{B.5})$$

Let us consider the simplest case $\mathbf{Q} = g \mathbb{I}_{n \times n}$. In this case, Eq. (B.5) is

$$\mathcal{M}^{mn} \{(\text{Tr} \mathcal{M})^2 - 2 \text{Tr}(\mathcal{M}^2)\} - \text{Tr}(\mathcal{M}^2) \{\text{Tr}(\mathcal{M}) \delta^{mn} - 2 \mathcal{M}^{mn}\} = 0, \quad (\text{B.6})$$

which is solved by

$$\mathcal{M} = \pm \mathbb{I}_{n \times n}, \quad (\text{B.7})$$

although only the positive sign gives a proper solution, consistent with the signature of the 11-dimensional spacetime.

These vacua will in general have a non-trivial φ , except in $d = 5$ in which the universal term of the potential does not depend on it and the vacuum solution is AdS_5 .

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